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Note on Solving for the Dynamics of the Universe

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ABSTRACT

In a recent article, Faraoni proposed an alternative procedure to solve the Friedman-Lemaitre-Robertson-Walker (FLRW) cosmological equations. The basic result of that paper was obtained long ago through a different approach, which seems to be little known and deserves closer attention due to its pedagogical interest. The broad importance of this method is readily recognized by examining some additional cases not considered by Faraoni. Its instructive potential for introductory courses on cosmology has been positively verified by the author in many lectures during the last decade.

Recently, Faraoni[1] rediscussed the cosmological solutions for a relativistic simple fluid in the framework of FLRW models. In this note we present a variant of the method proposed by Faraoni, which seems to be more general and quite interesting from a pedagogical viewpoint. Although little known, this approach was proposed long ago [2], and its pedagogical efficiency has also been verified many times in the last decade. In what follows, after a comparison with the results presented by Faraoni, we stress the generality of this procedure by discussing a class of solutions with a cosmological constant. For the sake of completness and further reference, as well as to consider some subtleties not addressed in Faraoni's work, we first summarize the basics of this problem.

The spacetime metric takes the following form ($c = 1$)

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \right) \tag{1}$$

where $a(t)$ is the scale factor, and $K = 0, \pm 1$ is the curvature parameter. In such a background, the Einstein field equations for a relativistic simple fluid can be written as [1-5]

$$8\pi G\rho = 3\frac{\dot{a}^2}{a^2} + 3\frac{K}{a^2} \ , \tag{2}$$

$$8\pi Gp = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} \ , \tag{3}$$

where ρ and p are the energy density and pressure, respectively. In this system there are three unknown quantities, namely $a(t)$, $\rho(t)$ and $p(t)$ and only two independent equations. Thus, in order to solve it, an additional constraint is required. In the cosmological framework, it is usually assumed that the matter content obeys the barotropic γ-law equation of state

$$p = (\gamma - 1)\rho \ , \qquad \gamma \in [0, 2] \quad . \tag{4}$$

This happens because such an expression provides a rather simple description for several cases of physical interest [6]: (i) dust or pressureless matter ($\gamma = 1$), (ii) radiation ($\gamma = 4/3$), (iii) vacuum ($\gamma = 0$), and (iv) Zeldovich's stiff matter ($\gamma = 2$). Naturally, the above assumption does not imply that the $\gamma - parameter$ remains constant in the course of the cosmological evolution. In the standard hot big bang description, the Universe underwent a transition from a primordial radiation phase ($\gamma = 4/3$) to a matter-dominated epoch ($\gamma = 1$) during the late stages of its evolution.

The cosmic dynamics is determined by combining the above set of equations. In principle, the corresponding dynamic behavior must be heavily dependent on the choice of the two free parameters: (i) the curvature parameter K, and (ii) the equation of state parameter γ. As one may check, the evolution of the scale function is driven by the second order differential equation[1, 2]

$$a\ddot{a} + \Delta\dot{a}^2 + \Delta K = 0 \quad , \tag{5}$$

where the parameter Δ (c in Faraoni's notation) is a function of γ

$$\Delta = \frac{3\gamma - 2}{2} \quad . \tag{6}$$

The first integral of (5) is

$$\dot{a}^2 = \left(\frac{a_o}{a}\right)^{2\Delta} - K \quad , \tag{7}$$

where the convenient integration constant, ${a_o}^{2\Delta}$, requires that all curves described by the general solution share the point $a = a_o$ at some instant of the cosmological time. Furthermore, the derivative $\dot{a}(a_o) = \sqrt{1-K}$ is the same for all values of γ, and indicates the existence of an extremum for closed models. We see from (5) and (6) that this extremum is a maximum if $\gamma > \frac{2}{3}$, a minimum if $\gamma < \frac{2}{3}$, or a stationary point if $\gamma = \frac{2}{3}$ [7]. In addition, equation (2) tells us that the energy density

$$\rho = \frac{3}{{a_o}^2} \left(\frac{a_o}{a}\right)^{3\gamma} \quad , \tag{8}$$

is positive definite, as should be expected for a physical fluid. Note also that for $K = 0$ the first integral (7) has the following solution

$$a(t) = a_o \left[1 + (\Delta + 1)(t - t_o)/a_o\right]^{\frac{1}{\Delta+1}} , \tag{9}$$

or equivalently,

$$a(t) = a_o \left[1 + \frac{3\gamma}{2}(\frac{t - t_o}{a_o})\right]^{\frac{2}{3\gamma}} , \tag{10}$$

where t_o is an arbitrary time scale. In particular, by choosing $t_o = 2a_o/3\gamma$ one recovers the restricted form of the flat solutions, namely, $a(t) = a_o(t/t_o)^{\frac{2}{3\gamma}}$. In addition, in the limit $\gamma \to 0$, the above expression reduces to $a(t) \sim e^{H_o t}$, which is the de Sitter flat solution ($H_o = a_o^{-1}$).

At this point we remark that the aim of Faraoni's paper is to obtain the general solution of the first integral (7), or equivalently of (5). This can be accomplished by using the conformal time η, instead of the cosmological or physical time ($dt = a(\eta)d\eta$). In this case, the metric (1) and equation (5) takes the forms below

$$ds^2 = a(\eta)^2 \left(d\eta^2 - \frac{dr^2}{1 - Kr^2} - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2\right) \tag{11}$$

$$aa'' + (\Delta - 1)a'^2 + \Delta K a^2 = 0 \quad , \tag{12}$$

where the prime denotes derivative with respect to conformal time. Now, instead of using Faraoni's transformation, $u = a'/a$, which leads to Ricatti's differential equation (see section 3 of [1]), we employ the auxiliary scale factor [2]

$$Z(\eta) = a^{\Delta}, \quad if \quad \Delta \neq 0 \tag{13}$$

$$Z(\eta) = \ln a, \quad if \;\; \Delta = 0 \quad , \tag{14}$$

to obtain, respectively,

$$Z'' + K\Delta^2 Z = 0, \quad if \quad \Delta \neq 0 \tag{15}$$

$$Z'' = 0, \quad if \;\; \Delta = 0 \quad . \tag{16}$$

Equation (15) is identical to equation (3.7) in Faraoni's paper. Note that it reduces to (16) in the limiting case $\Delta = 0$. Therefore, it can be considered the general equation of motion for any value of Δ. As remarked before, it depends strongly on the pair of parameters (γ, K). The physical meaning of (15) is apparent: It describes the classical motion of a particle subject to a linear force. This force is of restoring or repulsive type depending on the sign of the curvature parameter.

Closed models (K=1) are, for any value of $\Delta \neq 0$, analogous to harmonic simple oscillators (HSO). Therefore, the cosmic dynamics in this case is similar to a spring-mass system where the spring constant is determined by the γ-parameter defining the equation of state obeyed by the cosmic fluid (see

Eq.(6)). It is worth noticing that for positive values of Δ, that is, $\gamma > 2/3$, this general and intermittent oscillatory motion between both singularities ("big-bang" and "big-crunch") reinforces the connection with the idea of a pulsating Universe. From a pedagogical viewpoint, these harmonic solutions for closed models are more enlightening than the cycloidal parametric solutions usually presented in textbooks for particular values of γ [3-5]. It is actually surprising that the dynamics of closed universes might be reduced to that of a simple harmonic oscillator, a key system for analytical and algebraic calculations in many disparate branches of physics. It also leads naturally to the question whether our Universe is the largest clock, that is, a global oscillator. Indeed, although the current literature (including many textbooks) admits only a unique incomplete cycle to describe the observed Universe, the existence of pulsating solutions fascinate many philosophers and cosmologists since they are associated with the old concept of an ethernal return.

It is also worth mentioning that if the Universe is spatially flat ($K = 0$), equation (15) implies that the system behaves like a free particle, and the same happens if $\Delta = 0$. In the later case, this effective free particle behavior holds regardless of the curvature parameter. For hyperbolic spacetimes ($K = -1$), the system behaves like an "anti-oscillator", that is, a particle subject to a repulsive force proportional to the distance.

Now, recalling the elementary mathematical identities:

$$\lim_{\alpha \to 0} \frac{sin \alpha x}{\alpha} = x \tag{17}$$

$$sin(ix) = isinhx \quad , \tag{18}$$

the unified solution of (15) can be written as

$$Z = \frac{Z_o}{\sqrt{K}} sin\sqrt{K} \left[|\Delta| (\eta + \delta) \right] \tag{19}$$

where Z_o and δ are integration constants. Note also that using the transformation (13), the first integral (7) may be translated into the energy equation

$$\frac{1}{2}Z'^2 + \frac{1}{2}K\Delta^2 Z^2 = \frac{1}{2}\Delta^2 {a_o}^{\Delta} \tag{20}$$

which determines $Z_o = {a_o}^{\Delta}$.

By choosing $\delta = 0$, the general parametric solution relating the scale factor and the cosmological time, $dt = a(\eta)d\eta$, is given by

$$a(\eta) = a_o \left(\frac{sin\sqrt{K}|\Delta|\eta}{\sqrt{K}}\right)^{\frac{1}{\Delta}} , \tag{21}$$

$$t(\eta) = a_o \int \left(\frac{sin\sqrt{K}|\Delta|\eta}{\sqrt{K}}\right)^{\frac{1}{\Delta}} + \text{constant} \quad . \tag{22}$$

From identity (18) these solutions for elliptic and hyperbolic models read:

$K = 1$

$$a(\eta) = a_o \, (sin|\Delta|\eta)^{\frac{1}{\Delta}} \tag{23}$$

$$t(\eta) = a_o \int (sin|\Delta|\eta)^{\frac{1}{\Delta}} + \text{constant} \tag{24}$$

$K = -1$

$$a(\eta) = a_o \, (sinh|\Delta|\eta)^{\frac{1}{\Delta}} \tag{25}$$

$$t(\eta) = a_o \int (sinh|\Delta|\eta)^{\frac{1}{\Delta}} + \text{constant} \tag{26}$$

where the integrals (24) and (26) for $t(\eta)$ may be represented in terms of hypergeometric Gaussian functions [2]. As one may check using (17), the integral for $K = 0$ is trivial, and the parametric solutions may readily be inverted to give the scale factor as a function of the cosmological time.

It is also interesting to show that the above method based on the transforming equations (13) and (14) is also convenient when new ingredients are considered such as the presence of a cosmological Λ-term [8]. In this case, the EFE equations can be written as

$$8\pi G\rho + \Lambda = 3\frac{\dot{a}^2}{a^2} + 3\frac{K}{a^2} \ , \tag{27}$$

$$8\pi G p - \Lambda = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} \ , \tag{28}$$

where Λ is the cosmological constant. Now, using (4), one may see that the scale factor satisfies the generalized FLRW equation

$$a\ddot{a} + \Delta\dot{a}^2 + \Delta K - \frac{1}{3}\Delta(\Delta+1)\Lambda a^2 = 0 \quad . \tag{29}$$

Following the same steps of the case with no Λ, instead of (12), we obtain in the conformal time

$$aa'' + (\Delta - 1)a'^2 + \Delta K a^2 - \frac{1}{3}\Delta(\Delta+1)\Lambda a^4 = 0 \quad , \tag{30}$$

and transforming to the auxiliary scale factor through (13) results

$$Z'' + K\Delta^2 Z - \frac{1}{3}\Delta(\Delta+1)\Lambda Z^{\frac{\Delta+2}{\Delta}} = 0 \quad . \tag{31}$$

As expected, if $\Lambda = 0$, equations (29), (30) and (31) reduce to (5), (12) and (15), respectively.

Physically, the equation of motion (31) means that closed universes with cosmological constant evolve like anharmonic or non-linear oscillators [10]. The anharmonic contribution to the HSO is proportional to the cosmological Λ-term, and its power index depends uniquely on the equation of state γ-parameter. Note that the method provides an exact non-linear description. In other words, the anharmonic term does not appear due to the retention of higher order terms in some particular perturbative scheme.

Finally, we remark that the unidimensional equation of motion (5), as well as its generalized version with a cosmological constant are endowed with an obvious mechanical analogy which can be described within the classical Lagrangian formalism. It has been shown that a particle subject to a potential $V(a) = \alpha a^{-n} - \frac{1}{6}\Lambda a^2$, where α and n are constants, may satisfy the equation of motion (29). More precisely, this happens if $n = 2\Delta$, and the energy is proportional to the curvature, $2E = -mK$, where m is the mass of the test particle [9]. Naturally, the same method of solution, shown here for the relativistic case, may also be applied to this classical Lagrangian approach. The author has explored both approaches in many introductory courses on cosmology, and verified the nice pedagogical impact on the students.

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